AN INVESTIGATION

OF THE

ONE-HINGED STEEL ARCH

AND ITS

COMPARISON WITH OTHER TYPES

A THESIS

presented to the

Faculty of the Graduate School

Cornell University

for the degree of

Doctor of Philosophy

by

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AN INVESTIGATION OF THE ONE-HINGED ARCH AND ITS COMPARISON WITH OTHER TYPES

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PREFACE

During the last thirty years much study has been given to the design and construction of steel arches by American engineers and investigators. Two-hinged and three-hinged arches seem to have met the greatest favor, while few no-hinged steel arches have been built in this country. One-hinged arches are practically unknown in America, although a few have been successfully constructed on the European continent. After considerable study, the author has been in doubt of the practicability and value of the one-hinged steel arch. He could see no logical or conspicuous reason against its adoption. A number of writers attack it bitterly while enumerating a few disadvantages, but fail to demonstrate or justify their statements. Others deem it unnecessary to give a full treatment, because it has not come into popular use. The deficiency of theoretical knowledge may be the reason why the building of one-hinged steel arches has been attempted so rarely. With this idea in mind, the author has undertaken a special investigation of the one-hinged steel arch.

While this work is undertaken purely for the purpose of discovering and publishing some unknown facts, some contributions made by previous investigators will be mentioned. One of the noblest and most remarkable preliminary designs which have ever been made in bridge engineering was contributed by Charles Worthington in competition with other designs for the famous Quebec Bridge. It was a one-hinged steel arch with a span of 1,800 feet, more than twice as long as that of any arch bridge then existing in the world. One of the most noted American bridge engineers. Dr. B. A. L. Waddell, in his book called "Bridge Engineering," praises the work of Mr. Worthington and calls the design ingenious and quite feasible. It is to be regretted that the scheme was not accepted by the Canadian Government and thus the complete treatment of the theoretical and practical problems involved was prevented. The work gives some indication of what could be done with the one-hinged arch and done economically. Dr. Waddell's comment is valuable on account of his extensive experience and gives promise for the future. In view of these facts, the author felt encouraged to carry on this investigation.

Special emphasis is laid upon two points, first, to make an extensive study of its behavior in carrying the load; and second, to reveal its characteristics by critical comparisons with other types of arches. It is his hope that the work may be of value to the engineering profession in the future, if not at present. Since so little has been written in engineering literature upon the subject, it is hoped that every bit of the original work in this thesis will appeal to the sympathetic interest of engineers and investigators. Acknowledgment is due to Professor Henry S. Jacoby, chairman of the special committee in charge of the author's graduate work at Cornell University, for his aid and helpful suggestions.

Reactions Solved by the Author's Method of Symmetrical Deflection Equations.

Two kinds of beams, either straight or curved, are used in bridge building; those which are statically determinate, and those statically indeterminate. For the solution of beams of the first class, three statical equations are available:—

$$\Sigma V = 0 \quad \text{(a)}$$

$$\Sigma H = 0 \quad \text{(b)}$$

$$\Sigma M = 0 \quad \text{(c)}$$

For the solution of beams of the second class, three elastic conditions are available, besides the statical conditions. These are:—

$$\Delta x = \int_a^b \frac{Myds}{EI} \quad \text{(d)}$$

$$\Delta y = \int_a^b \frac{Mxds}{EI} \quad \text{(e)}$$

$$\Delta \theta = \int_a^b \frac{Mds}{EI} \quad \text{(f)}$$

where Δx is the horizontal deflection, Δy the vertical deflection, and $\Delta \theta$ the change of the slope between any two points on the beam, E the modulus of elasticity of its material, I the moment of inertia of a section, and M the moment at any point between a and b, where I is taken. By means of these six fundamental equations, all beams can be solved. (For the derivation of formulas (d), (e) and (f) see standard books on Mechanics). The

author has used these equations in deriving general formulas of reactions for no-, one-, two-, and three-hinged arches.

Let an arch-rib with a span l and rise h be fixed at two ends a and b, and hinged at the crown c and be subject to a vertical load P at a distance kl from the left end. (Fig. 1). The load is sustained by the reactions H_1, V_1 and M_1 at the left support, and by H_2, V_2 and M_2 at the right support. There are six unknowns and six conditions are required for its solution. The three statical equations furnish three conditions, while the hinge at the crown insures that the moment at that point is zero. It remains for us to find two more elastic conditions. In other words, we must choose any two equations from (d), (e) and (f) and apply them to the case of the one-hinged arch.

In order to simplify the algebraic work, the following notations are proposed by the author:—

$$p=\int_0^{\frac{1}{2}l}\frac{x\,ds}{I} \qquad p_1=\int_0^{kl}\frac{x\,ds}{I}$$

$$q=\int_0^{\frac{1}{2}l}\frac{y\,ds}{I} \qquad q_1=\int_0^{kl}\frac{y\,ds}{I}$$

$$r=\int_0^{\frac{1}{2}l}\frac{xy\,ds}{I} \qquad r_1=\int_0^{kl}\frac{xy\,ds}{I}$$

$$s=\int_0^{\frac{1}{2}l}\frac{x^2ds}{I} \qquad s_1=\int_0^{kl}\frac{x^2ds}{I}$$

$$t=\int_0^{\frac{1}{2}l}\frac{y^2ds}{I} \qquad t_1=\int_0^{kl}\frac{y^2ds}{I}$$

$$p_2=\int_{kl}^{\frac{1}{2}l}\frac{x\,ds}{I} \qquad p=p_1+p_2$$

$$q_2=\int_{kl}^{\frac{1}{2}l}\frac{y\,ds}{I} \qquad q=q_1+q_2$$

$$r_2=\int_{kl}^{\frac{1}{2}l}\frac{xy\,ds}{I} \qquad r=r_1+r_2$$

$$s_2=\int_{kl}^{\frac{1}{2}l}\frac{x^2ds}{I} \qquad s=s_1+s_2$$

$$t_2=\int_{kl}^{\frac{1}{2}l}\frac{y^2ds}{I} \qquad t=t_1+t_2$$

The hinge at the crown breaks the continuity of the beam, hence a special method must be used in applying the two elastic conditions. M. A. Howe in his book called "A Treatise on Arches" succeeds in reducing the required elastic conditions into one by means of symmetrical loading. But the method is rather lengthy, and the formulas obtained rather complicated. By means of symmetrical deflection equations, the problem is easily solved by the author, with the final formulas expressed in a very simple form. Let us assume that the arch-rib is divided into two parts at the crown by an imaginary vertical line at the crown. (Fig. 1). Formulas (d) and (e) are used in finding the vertical and horizontal deflections between a and c and between b and c. As the supports are actually fixed, the vertical deflection between a and c must be equal to that between b and c; thus giving the first condition. Also, the horizontal deflections between a and c and between b and c must be equal in magnitude and opposite in direction, because what is lengthened in one-half of the rib must be shortened in the other. Thus, we obtain the second condition. For the left half of the rib with the origin at a the moment at any point between a and c is

$$M=M_1+V_1x-Hy-P(x-kl) \quad \ldots(1)$$

For the right half of the rib with the origin at b the moment at any point between b and c is

$$M=M_2+V_2x-Hy \quad \ldots\ldots(2)$$

Substituting (1) in (d) and (e) and replacing the integral forms with the notation above given, we have the horizontal and vertical deflections between a and c respectively,

$$E\Delta x=M_1q+V_1r-Ht-P(r_2-klq_2) \quad \ldots(3)$$
$$E\Delta y=M_1p+V_1s-Hr-P(s_2-klp_2) \quad \ldots(4)$$

Substituting (2) in the same formulas and making the same substitutions for the integral forms, we have

$$E\Delta x=M_2q+V_2r-Ht \quad \ldots(5)$$
$$E\Delta y=M_2p+V_2s-Hr \quad \ldots(6)$$

where the modulus of elasticity is assumed to be constant. Equating (3) to (5) and (4) to (6) as above mentioned, we have

$$M_1p+V_1s-Hr-P(s_2-klp_2)=M_2p+V_2s-Hr \quad (7)$$
$$M_1q+V_1r-Ht-P(r_2-klq_2)=-(M_2q+V_2r-Ht) \quad (8)$$

These two equations together with the three statical equations and the equation furnished by the hinge at the crown as shown below are sufficient to solve all the unknowns.

$$M_1+V_1l-Pl(l-k)-M_2=0 \quad \ldots(9)$$

$$M_1+V_1\frac{l}{2}-Hh-\frac{Pl}{2}(1-2k)=0 \quad \ldots(10)$$
$$V_1+V_2-P=0 \quad \ldots(11)$$
$$H_1-H_2=0 \quad \ldots(12)$$

Thus, by solving (7), (8), (9), (10), (11) and (12), we have

$$H_1 = -H_2 = P\frac{klq_1 - r_1}{2(hq-t)} \qquad \text{......(A)}$$

$$V_1 = P\frac{lp - klp_1 - s - s_2}{lp - 2s} \qquad \text{......(B)}$$

$$V_2 = P\frac{klp_1 - s_2}{lp - 2s} \qquad \text{......(C)}$$

$$M_1 = -V_1\frac{l}{2} + Hh + \frac{Pl}{2}(1-2k) \qquad \text{......(D)}$$

$$M_2 = M_1 + V_1 l - Pl(1-k) \qquad \text{......(E)}$$

It must be remembered that these formulas hold good only when the load is on the left of the crown. They are applicable to a one-hinged arch with any form of arch-ribs.

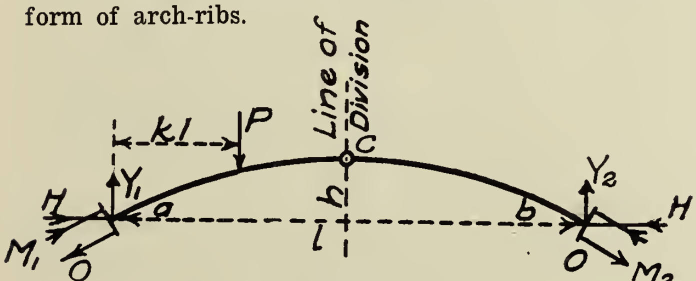


FIG. 1

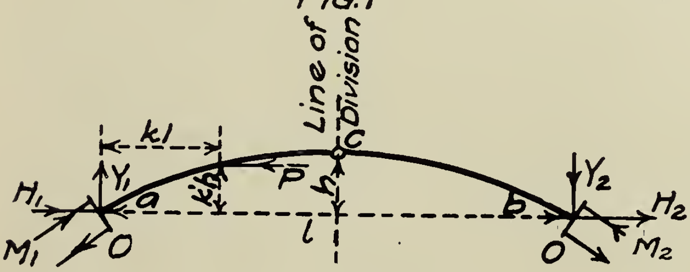


FIG. 2

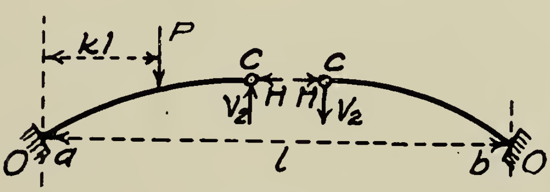


FIG. 3

E.J.S.

If instead of a vertical load P on the span, we place a horizontal load P at the same position acting towards the left and at a distance $k'h$ from the line ab, (Fig. 2) the problem can be solved in a similar way. In this case the direction of H_2 and V_2 are reversed in comparison with those under the vertical loading. The following formulas are derived for the horizontal load on the span with the same process:

$$H_1 = P\frac{2hq - hk'q_1 - t - t}{2(hq-t)} \qquad \text{....(F)}$$

$$H_2 = P\frac{k'hq_1 - t_1}{2(hq-t)} \qquad \text{....(G)}$$

$$V_1 = -V_2 = P\frac{r_1 - k'hp_1}{2s - lp} \qquad \text{....(H)}$$

$$M_1 = -\frac{V_1 l}{2} + H_1 h - Ph(1-k') \qquad \text{....(I)}$$

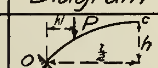
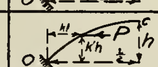
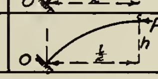
$$M_2 = M_1 + V_1 l - Pk'h \qquad \text{....(J)}$$

These formulas again hold good only when the load is on the left of the crown with its direction toward the left. They are also applicable to any form of arch-ribs.

Reactions Solved by Author's Cantilever Method

Another interesting method is derived by the author in securing the two elastic equations for solving the reactions of a one-hinged arch. It is called the cantilever method, because the main feature lies in separating the arch into two separate cantilevers free at the crown and fixed at the supports. The vertical and horizontal deflections at the free end of each cantilever are then found and equated so as to furnish the two necessary equations. The method is very simple, because the deflections at the free ends of cantilevers can be easily obtained. Thus, the following deflection table includes all the data needed for the solution:

Table 1. Deflection Table for Curved Beams

Loading	Diagram	Δx at c	Δy at c
Vert. Load at kl from o		$-\frac{P}{E}(klq_1 - r_1)$	$-\frac{P}{E}(klp_1 - s_1)$
Hor. Load at kl from o		$+\frac{P}{E}(k'hq_1 - t_1)$	$+\frac{P}{E}(k'hp_1 - r)$
Vert. Load at free end		$-\frac{P}{E}(\frac{l}{2}q - r)$	$-\frac{P}{E}(\frac{l}{2}p - s)$
Hor. Load at free end		$+\frac{P}{E}(hq - t)$	$+\frac{P}{E}(hp - r)$

The expressions in this table can be easily derived by using formulas (d) and (e).

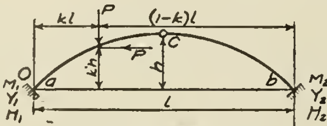
In order to find the elastic conditions, let us separate the arch at the crown. Since there is no load on the right half of the rib (See Fig. 3) and since the right reaction line must pass through the crown, the two forces H and V_2 must act at the free end of the right cantilever with their directions as shown. Therefore we can consider the left cantilever as subjected to two external forces H and V_2 at the free end. On the left half of the rib there is a load P. But since action must equal the reaction, the forces acting at the free end of the left cantilever must be H and V_2. Let us find the vertical and horizontal deflections between a and c when the loads H, V_2, and P are acting on the left cantilever. Evidently, the vertical and horizontal deflections at c are contributed by three factors, those due to P, H and V_2 respectively. From table 1 these factors can be summed up in the expressions,

$$E\Delta y = V_2\left(\frac{l}{2}p - s\right) + H(hp - r) - P(klp_1 - s_1) \qquad (13)$$

$$E\Delta x = V_2\left(\frac{l}{2}q - r\right) + H(hq - t) - P(klq_1 - r_1) \qquad (14)$$

Similarly, the vertical and the horizontal deflections at c of the right cantilever are contributed by two factors, those due to H and V_2. From table 1, we have the expressions,

Table 2—Reactions and End Moments of a One-Hinged Ar h with Various Forms of Arch-Ribs.



One-Hinged Arch

$p=\int_0^t \frac{x\,ds}{I}$ $\quad t_1=\int_0^{kl} \frac{y^2ds}{I}$

$p_1=\int_0^{kl} \frac{x\,ds}{I}$ $\quad t_2=\int_{kl}^{t} \frac{y^2ds}{I}$

$p_2=\int_{kl}^t \frac{x\,ds}{I}$ $\quad m=\sqrt{K-K^2}$

$q=\int_0^t \frac{y\,ds}{I}$ $\quad n=32-3\pi^2$

$q_1=\int_0^{kl} \frac{y\,ds}{I}$ $\quad o=3\pi-8$

$q_2=\int_{kl}^t \frac{y\,ds}{I}$ $\quad \alpha=\sin^{-1}(2k-1)$

$r=\int_0^t \frac{xy\,ds}{I}$ $\quad g=r-h$

$r_1=\int_0^{kl} \frac{xy\,ds}{I}$ $\quad \cos\alpha=\frac{l}{2r}$

$r_2=\int_{kl}^t \frac{xy\,ds}{I}$ $\quad r=\frac{4h^2+l^2}{8h}$

$r_4=\int_0^l \frac{xy\,ds}{I}$ $\quad I=I_0\sec\theta$

$s=\int_0^t \frac{x^2ds}{I}$ $\quad E=\text{constant}$

$s_1=\int_0^{kl} \frac{x^2ds}{I}$

$s_2=\int_{kl}^t \frac{x^2ds}{I}$

$t=\int_0^t \frac{y^2ds}{I}$

$K=24r^2(r-\frac{h}{2})(\frac{\pi}{2}-\alpha)+l(l^2-12r^2-6hg)$

$Q=12r^2l-l^3-24r^2g(\frac{\pi}{2}-\alpha)$

NOTE:
1 Notations as shown above.
2. Assumptions, $I=I_0\sec\theta$. E = constant, shear neglected
3. Formulae under Bending Moment hold good, when the load is on the left half of the span.

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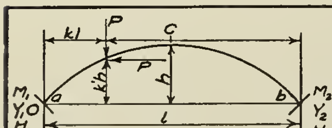
CURVE			GENERAL	PARABOLIC	ELLIPTICAL	SEGMENTAL CIRCULAR
BENDING MOMENTS	VERTICAL LOAD	H	$P\frac{klq_1-r_1}{2(hq-t)}$	$\frac{5}{2}\frac{Pl}{h}(2k^3-k^4)$	$\frac{Pl}{4ho}\left\{\begin{matrix}4(4k^2-4k+3)m+\\3(1-2k)(3\pi-2\alpha)\end{matrix}\right\}$	$\frac{P}{2K}\left\{\begin{matrix}8r^3\sin^3\alpha'+3r^2l(1-2k)(\sin 2\alpha'\\+2\alpha'-2\alpha)-3g^3-3l^2g(1-2k+4k^2)\end{matrix}\right\}$
		V_1	$P\frac{lp-klp_1-s-s_2}{lp-2s}$	$P(1-4k^3)$	$P(1-4k^3)$	$P(1-4k^3)$
		V_2	$P\frac{klp_1-s_1}{lp-2s}$	$4Pk^3$	$4Pk^3$	$4Pk^3$
		M_1	$-V_1\frac{l}{2}+Hh+\frac{Pl}{2}(1-2k)$	$-\frac{Pl}{2}(2k-14k^3+5k^4)$	$-\frac{V_1l}{2}+Hh+\frac{Pl}{2}(1-2k)$	$-\frac{V_1l}{2}+Hh+\frac{Pl}{2}(1-2k)$
		M_2	$M_1+V_1l-Pl(1-k)$	$\frac{Pl}{2}(6k^3-5k^4)$	$M_1+V_1l-Pl(1-k)$	$M_1+V_1l-Pl(1-k)$
	HORIZONTAL LOAD	H_1	$P\frac{2hq-k'hq_1-t-t_2}{2(hq-t)}$	$P(1-20k^3+40k^4-16k^5)$	$\frac{P}{o}\left\{\begin{matrix}o+4k(3-3k+2k^2)\\-3m(2\alpha-3\pi)\end{matrix}\right\}$	$P-H_2$
		H_2	$P\frac{k'hq_1-t_1}{2(hq-t)}$	$4P(5k^3-10k^4+4k^5)$	$P-H_1$	$\frac{P}{2K}\left\{\begin{matrix}6lg^2-2kl[12r^2-l^2(3-6k+4k^2)]\\-6r^2[\sin 2\alpha'+2(\alpha'-\alpha)][g+r\sin\alpha']\\+6lrg(1-4k)\sin\alpha'\end{matrix}\right\}$
		V_1	$P\frac{r_1-k'hp_1}{2s-lp}$	$\frac{8Ph}{l}(2k^3-3k^4)$	$\frac{Ph}{2l}\left\{\begin{matrix}4m(4k^2+2k+3)-\\6\alpha+9\pi\end{matrix}\right\}$	$P\left\{\begin{matrix}\sin^3\alpha'+\frac{3}{4}\sin 2\alpha'\\+6K^2\sin\alpha'-\frac{3}{2}\alpha'\end{matrix}\right\}$
		V_2	$-P\frac{r_1-k'hp_1}{2s-lp}$	$-\frac{8Ph}{l}(2k^3-3k^4)$	$-V_1$	$-V_1$
		M_1	$-\frac{V_1l}{2}+H_1h-Ph(1-k')$	$4Ph(k-k^2-7k^3+13k^4-4k^5)$	$-\frac{V_1l}{2}+H_1h-Ph(1-k')$	$-\frac{V_1l}{2}+H_1h-Ph(1-K')$
		M_2	$M_1+V_1l-Pk'h$	$-4Ph(3k^3-7k^4+4k^5)$	$M_1+V_1l-Pk'h$	$M_1+V_1l-Pk'h$
TEMPER-ATURE EFFECTS		H_1	$\pm\frac{Eet^\circ l}{2(hq-t)+\frac{lr_1^2}{I_0}}$	$\pm\frac{15Eet^\circ I_0}{2h^2+15r_1^2}$	$\pm\frac{12Eet^\circ I_0}{oh^2+12r_1^2}$	$\pm\frac{12Eet^\circ lI_0}{K+12lr_1^2}$
		M_1	$\mp\frac{Eet^\circ lh}{2(hq-t)+\frac{lr_1^2}{I_0}}$	$\mp\frac{15Eet^\circ I_0h}{2h^2+15r_1^2}$	$\mp\frac{12Eet^\circ I_0h}{oh^2+12r_1^2}$	$\mp\frac{12Eet^\circ lI_0h}{K+12lr_1^2}$
RIB SHORT-ENING		H_1	$-\frac{sl}{2(hq-t)}$	$-\frac{15sI_0}{2h^2}$	$-\frac{12sI_0}{oh^2}$	$-\frac{12slI_0}{K}$
		M_1	$+\frac{shl}{2(hq-t)}$	$+\frac{15sI_0}{2h}$	$-+\frac{12sI_0}{oh}$	$+\frac{12slhI_0}{K}$

Table 3—General Formulas of Reactions and End Moments of an Arch with the Regular Form of Arch Rib.



Arch with Regular Curves

$p=\int_0^t \frac{x\,ds}{I}$ $\quad r_4=\int_0^l \frac{xy\,ds}{I}$

$p_1=\int_0^{kl} \frac{x\,ds}{I}$ $\quad s=\int_0^t \frac{x^2ds}{I}$

$p_2=\int_{kl}^t \frac{x\,ds}{I}$ $\quad s_1=\int_0^{kl} \frac{x^2ds}{I}$

$q=\int_0^t \frac{y\,ds}{I}$ $\quad s_2=\int_{kl}^t \frac{x^2ds}{I}$

$q_1=\int_0^{kl} \frac{y\,ds}{I}$ $\quad t=\int_0^t \frac{y^2ds}{I}$

$q_2=\int_{kl}^t \frac{y\,ds}{I}$ $\quad t_1=\int_0^{kl} \frac{y^2ds}{I}$

$r=\int_0^t \frac{xy\,ds}{I}$ $\quad t_2=\int_{kl}^t \frac{y^2ds}{I}$

$r_1=\int_0^{kl} \frac{xy\,ds}{I}$ $\quad I=I_0\sec\theta$

$r_2=\int_{kl}^t \frac{xy\,ds}{I}$ $\quad E=\text{Constant}$

NOTE:
1. Formulae for one and three Hinged Arches under the term Bending Moment hold good, when the load is on the left of C.
2. Rise or Fall of Temperature has certain effect upon H of the three Hinged. The formula is not derived, as its effect upon the truss can best be found by the graphical method. (—)
3. Notations are shown above. r_1 = radius of Gyration. s = unit compressive stress in Rib-shortening Formulae. (average stress)

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ARCH			NO-HINGED	ONE-HINGED	TWO-HINGED	THREE-HINGED
BENDING MOMENTS	VERTICAL LOAD	H	$\frac{Pl}{2}\frac{(r_1+klq_2)-lq(k-k^2)}{lt-2q^2I_0}$	$P\frac{klq_1-r_1}{2(hq-t)}$	$P\frac{r_1+klq_2}{2t}$	$\frac{Pkl}{2h}$
		V_1	$\frac{M_2-M_1}{l}+P(1-k)$	$P\frac{lp-klp_1-s-s_2}{lp-2s}$	$P(1-k)$	$P(1-k)$
		V_2	$P-V_1$	$P\frac{klp_1-s_1}{lp-2s}$	Pk	Pk
		M_1	$\frac{2HI_0}{l_2}(4lq-3r_4)-Pkl(1-k)^2$	$-V_1\frac{l}{2}+Hh+\frac{Pl}{2}(1-2k)$	0	0
		M_2	$M_1+V_1l-Pl(1-k)$	$M_1+V_1l-Pl(1-k)$	0	0
	HORIZONTAL LOAD	H_1	$\frac{P}{2}\left\{\frac{l[t_2-k'hq_2]-q[2q_2-1+k'hl(1-2k)]}{lt-2q^2I_0}\right\}$	$P\frac{2hq-k'hq_1-t-t_2}{2(hq-t)}$	$\frac{P}{2}(1+\frac{t_2-k'hq_2}{t})$	$\frac{P}{2}(2-k')$
		H_2	$P-H_1$	$P\frac{k'hq_1-t_1}{2(hq-t)}$	$P-H_1$	$\frac{Pk'}{2}$
		V_1	$\frac{1}{l}(M_2-M_1+Pk'h)$	$P\frac{r_1-k'hp_1}{2s-lp}$	$\frac{Pk'h}{l}$	$\frac{Pk'h}{l}$
		V_2	$-V_1$	$-P\frac{r_1-k'hp_1}{2s-lp}$	$-\frac{Pk'h}{l}$	$-\frac{Pk'h}{l}$
		M_1	$\frac{2H_1}{l_2}(4lq-3r_4)+\frac{2P}{l_2}[3(r_4-r_1)-2l(2q-q_1)]+Pk'h(1-4k+3k^2)$	$-\frac{V_1l}{2}+H_1h-Ph(1-k')$	0	c
		M_2	$M_1+V_1l-Pk'h$	$M_1+V_1l-Pk'h$	0	0
TEMPER-ATURE EFFECTS		H_1	$\pm\frac{Eet^\circ l}{2t-\frac{4q^2I_0}{l}+\frac{lr_1^2}{I_0}}$	$\pm\frac{Eet^\circ l}{2(hq-t)+\frac{lr_1^2}{I_0}}$	$\pm\frac{Eet^\circ l}{2t+\frac{lr_1^2}{I_0}}$	—
		M_1	$\mp\frac{2Eet^\circ q}{2t-\frac{4q^2I_0}{l}+\frac{lr_1^2}{I_0}}$	$\mp\frac{Eet^\circ lh}{2(hq-t)+\frac{lr_1^2}{I_0}}$	0	0
RIB SHORT-ENING		H_1	$-\frac{sl}{2t-\frac{4q^2I_0}{l}}$	$-\frac{sl}{2(hq-t)}$	$-\frac{sl}{2t}$	0
		M_1	$+\frac{sq}{t-\frac{2q^2I_0}{l}}$	$+\frac{shl}{2(hq-t)}$	0	0

$$E\Delta y = -V_2\left(\frac{l}{2}p-s\right) + H(hp-r) \quad \ldots(15)$$
$$E\Delta x = -V_2\left(\frac{l}{2}q-r\right) + H(hq-t) \quad \ldots(16)$$

The value of E is assumed to be constant in all of these expressions. Equating the vertical deflections from each half of the rib, we have

$$V_2\left(\frac{l}{2}q-r\right) + H(hq-t) - P(klq_1-r_1)$$
$$= -V_2\left(\frac{l}{2}p-s\right) + H(hp-r) \quad \ldots\ldots(17)$$

from which the formula for V_2 can be obtained directly. Equating the horizontal deflections we have

$$V_2\left(\frac{l}{2}p-s\right) + H(hp-r) - P(klp_1-s_1)$$
$$= V_2\left(\frac{l}{2}p-r\right) - H(hq-t) \quad \ldots\ldots(18)$$

from which the formula for H can be obtained directly. From the statical conditions and the condition furnished by the hinge at the crown, we can obtain all necessary formulas for the reactions and the bending moments at the ends.

In the case of the horizontal loading, the above method applies equally well. The expressions for the vertical and horizontal deflections at the free end of a cantilever under a horizontal load P are also given in table 1.

Influence of Temperature

Changes in temperature cause changes in the values of M_1 and H_1 but do not affect the vertical reaction V_1. It is usually specified that an arch shall be designed to be subject to a certain variation in temperature from a standard value. Let Σ be the coefficient of expansion and t° the rise of temperature, then the span length will be increased by $\Sigma t^\circ l$ provided one end is free to move. As both ends are fixed in position when the supports do not yield, equal and opposite reactions and end moments are produced. The values of H_1 and M_1 must be such as to prevent the horizontal displacement $\Sigma t^\circ l$, which is due to the effect of bending and thrust. The horizontal deflection due to bending and thrust are given by the following expressions:

$$\Delta x_1 = \int_0^l \frac{Myds}{EI} \quad \ldots\ldots\ldots(19)$$

$$\Delta x_2 = \frac{H_1 l}{AE} \quad \ldots\ldots\ldots(20)$$

in which A is the area of cross-section. Equating the deflections, and

substituting $H_1(h\text{-}y)$ for M and solving for H_1 and M_1, we have

$$H_1 = \pm\frac{Eet^\circ l}{2(hq-t)+\frac{lr_1^2}{I_o}} \quad \ldots\ldots\ldots(K)$$

$$M_1 = \mp\frac{Eet^\circ hl}{2(hq-t)+\frac{lr_1^2}{I_o}} \quad \ldots\ldots\ldots(L)$$

where Io is the moment of inertia of the section at the crown and r_1 is the radius of gyration of the same section. For a rise in temperature use the positive sign for H_1 and a negative sign for M_1. For a fall in temperature, the reverse is true.

Rib-Shortening

The direct effect of the thrust along the axis is to shorten the axis of the arch-rib. It would also shorten the span provided one end were free to move, but as this is not the case it will develop equal and opposite negative reactions H_1 and positive moments M_1. The effect of displacement due to H_1 and M_1 must equal that due to rib-shortening. Let us use S as the average compressive stress on the rib. The shortening of the span is given by

$$\Delta = \frac{Sl}{E} \quad \ldots\ldots\ldots\ldots\ldots(21)$$

which must be equal to

$$\Delta = \int_0^l \frac{Myds}{EI} \quad \ldots\ldots\ldots\ldots(22)$$

Equating (18) to (19), substituting $H_1(h\text{-}y)$ for M in the equation, and solving for H_1, we have

$$H_1 = -\frac{Sl}{2(hq-t)} \quad \ldots\ldots\ldots(M)$$

$$M_1 = \frac{Shl}{2(hq-t)} \quad \ldots\ldots\ldots(N)$$

Application to Parabolic, Segmental Circular, and Elliptical One-Hinged Arches

The general formulas derived above have been applied to three forms of arch-ribs with parabolic, segmental circular and elliptical curves respectively. The equations of these curves all have their origin at the left support a, with the X-axis passing through the supports a and b. The equations of these curves are as follows:

$$y = 4h\left(\frac{x}{l}-\frac{x^2}{l^2}\right) \quad .(23)$$

$$(y+r-h)^2 = 2r\left(x+r-\frac{l}{2}\right)-\left(x+r-\frac{l}{2}\right)^2 \quad .(24)$$

$$\frac{\left(x-\frac{l}{2}\right)^2}{\left(\frac{l}{2}\right)^2}+\frac{y^2}{h}=1 \quad .(25)$$

An important assumption is made in applying the general formulas, that is $I{=}Io\ sec\ \theta$ where I is the moment of inertia of the cross-section at any point and

o the angle of inclination of the axis of rib at the point where I is taken. This assumption simplifies the work materially and should be close enough for deriving the formulas to be used in the preliminary designs. Also, $ds = \sec o\,dx$, while the modulus of elasticity is assumed to be constant in the general formulas. Table 2 gives a comparison of the formulas for the reactions and end moments of the one-hinged arches with parabolic, elliptical and segmental circular arch ribs. These formulas have been derived after a tremendous amount of work. Formulas for temperature effects and rib-shortening are also given in the table.

The Reaction Locus

Fig. 1 shows a one-hinged arch under a vertical load P. As the ends are fixed, it is unknown where the reaction lines must pass the supports. Let us assume that they pass the left and the right supports at points distant y_1 and y_2 below and above the points of support respectively. The right reaction line must pass through the center hinge. By similar triangles, we have the relation

$$\frac{q+y_1}{kl} = \frac{V_1}{H}$$ in which q is the ordinate to the curve at any point.

But $y_1 = -\frac{M_1}{H_1}$, therefore

$$q = \frac{klV_1 + M_1}{H_1} \qquad \text{(O)}$$

which is a general formula of the reaction locus of the one-hinged and the no-hinged arches applicable to any form of arch-rib. It is not applicable when the load is on the right half of the arch rib. The reaction locus must be symmetrical about the center, so it is not necessary to derive the formulas for the load on the right of the span. For parabolic, elliptical and segmental circular arches we have by substitution the following formulas:

$$q = \frac{h(14-13k)}{5(2-k)} \qquad q = \frac{h(1+13k)}{5(1+k)} \qquad \text{(P)}$$

for the load on the left and right of the crown hinge respectively.

Equation Q*

Equation R†

Formulas for Reactions of Three—,Two—And No-Hinged Arches

General formulas for reactions of three-, two-, and no-hinged arches have been derived in order to obtain a comparison between them. They have also been applied to these types of arches with parabolic, elliptical, and segmental circular ribs. Table 3 shows a comparison of the general formulas of the four types of arches, while those for a comparison of the reactions of parabolic, segmental circular and elliptical forms of arches are not here reproduced.

The Envelop of Reaction Lines

The position of live load for the maximum positive and negative moments and the maximum positive and negative shears of the one-hinged arch can be found graphically by means of the reaction locus and the envelop of the reaction lines. Since the ends of the one-hinged arch are fixed, we cannot tell where the reaction lines pass through the support for a load at a certain point of the span. By means of the reaction envelop and the reaction locus, this can be easily done by drawing the tangent line to the reaction envelop from the point where the load line intersects the reaction locus. The equation of the reaction envelop of the one-hinged arch was found by the author to be an equation of the third degree. To plot the curve from the equation involves a tremendous amount of labor. It is easier to draw the reaction envelop from the computed values of ordinates than to plot it from the equation. The envelop passes through the hinge and intersects the Y-axis at a certain distance below the origin. To determine the points of division of the live load for the maximum positive and negative moments, tangent lines are drawn to the envelop from the point on the arch-rib we wish to investigate. The intersection of the tangent lines with the reaction locus gives the required points of division. To determine the points of division for the shear, a tangent line to the envelop is drawn perpendicular to the normal line of the section. The intersection of this tangent line with the reaction locus forms one division point, while the section itself is another division point.

Study On Designs

The design of a one-hinged arch can only be made with a series of approximations. Stresses due to temperature and rib-shortening play an important

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* $$q = h\left[1 + 8(3\pi - 8)\frac{(1-2k)k^3}{4(4k^2-4k+3)\sqrt{k-k^2} + 3(1-2k)(3\pi - 2\alpha)}\right] \qquad \text{(Q)}$$

in which $\alpha = \cos^{-1}\frac{l}{2r}$

† $$q = h\left[1 + \frac{4lk^4(1-2k)}{h[8r^3\sin^3\alpha' + 3r^2l(1-2k)(\sin 2\alpha' + 2\alpha - 2\alpha') - 8g^3 - 3l^2g(1-2k+4k^2)]}\right] \qquad \text{(R)}$$

in which $g = r - h \qquad \alpha' = \sin^{-1}(2k-1)$

part. These can not be exactly determined without knowing the area of the cross-section of the arch-rib. Yet the latter again depends upon the total stresses to be carried. The method of trial is the only way to secure the right section for the one-hinged arch-rib. Three comparative designs were made by the author in order to study their comparative merits: (1) The moment of inertia at any section of the arch-rib is assumed to vary as the secant of the angle of inclination of the section, with the effect of rib-shortening neglected; (2) Same assumption as in the first case, with the effect of rib-shortening included; and (3) the true moment of inertia of the section is used.

A one-hinged arch of twenty panels, with a span of 258 feet and a rise of 26′ was adopted for investigation. The dead load is assumed 59 kips and the live load 18.5 kips per panel load. The depth of the rib at the center is assumed 5 feet. An allowance of 75° F is made for the rise or fall of temperature. The Modulus of elasticity is taken as 26,000 kips and the unit-stress for the gross section of the flanges, 15 kips. The arch chosen for investigation has been greatly studied by former graduate students of Professor H. S. Jacoby when containing three, two or no hinges. The reason for using this arch by the author is, of course, to compare the design of the one-hinged arch with that of the other types.

Study On Deflections

The deflections of the one-hinged arch were studied in three ways: (1) Vertical and horizontal deflections under the vertical loading; (2) vertical and horizontal deflection under the horizontal loading; and (3) maximum and minimum deflections. The same arch used in the design was used for the study of deflections. The deflections of this arch having three, two and no hinges respectively were investigated by Dr. P. H. Chen, a graduate of Cornell University. It gives a good opportunity for critical comparisons. (See The Cornell Civil Engineer, Vol. 26, Page 229).

RESULTS OF INVESTIGATION

Discussion On Formulas

So far as the formulas for the reactions of the one-hinged arch are concerned, those for the parabolic ribs are the simplest in form, while those for the segmental circular form require a vast amount of labor for their derivation and application. Formulas for the elliptical form are intermediate in these respects.

Neglecting the effect of the axial thrust and using a ratio of the rise to the span for a segmental circular arch equal to 1-8, the relative effect of temperature upon the three forms of the one-hinged arch is as follows:

	Parabolic	Elliptical	Seg. Circular
Coefficient of $Et t^o I_o/h^2$	7.5	8.4	6.0
Ratio	125	140	100

The temperature effect in the parabolic form is 125% as great, and that in the elliptical form 140% as great as that for the segmental circular form. The effect upon the elliptical form is 112% as great as that upon the parabolic form. Thus, the temperature effect upon the elliptical form is the largest, while that upon the circular form is the smallest. It is to be noted that in the segmental form the ratio of the rise to the span is assumed as one-eighth. This is only a particular case. If we assumed another ratio, the results would be different.

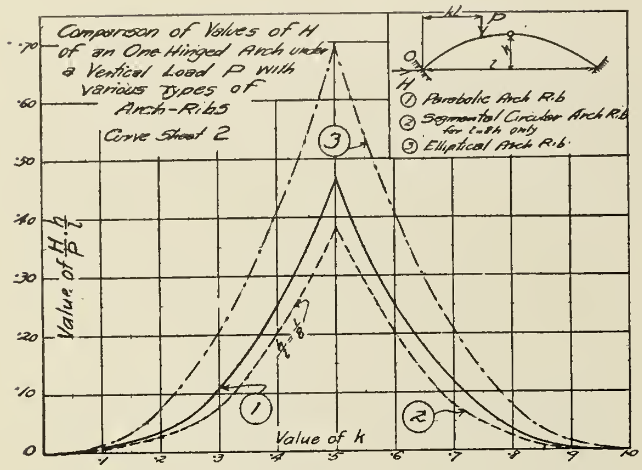
The relative effects of rib-shortening upon the horizontal reaction and the end moments of the three forms of one-hinged arches have the same relations as the temperature effects.

There is an interesting fact about the number of equations required to analyze no-, one-, two-, and three-hinged arches. Three-hinged arches require three conditions; two-hinged arches, four; one-hinged arches, five; and no-hinged arches, six. Thus, we see that the more hinges an arch has, the less are the number of the conditions required to solve its unknowns. In addition to the three statical conditions, the three-hinged arch requires no-elastic condition for its solution; the two-hinged arch requires one; the one-hinged arch, two; while the ne-hinged arch requires three. Therefore the three-hinged arch is said to be statically determinate.

The formulas for reactions of three-hinged arches are exact, while those for no-, one-, and two-hinged arches are subject to many imperfections and assumptions. First of all, the elastic limit is not assumed to be passed. If very large loads should ever be applied which cause the stresses to exceed the elastic limit of the material in any member, the theory of elasticity fails and it is impossible to predict the degree of safety of the structure. The moment of inertia of any section of the rib is assumed to vary as the secant of the angle of the inclination of the arch-rib. The modulus of elasticity is assumed to be constant throughout the span. Shear is neglected in the derivation of the formulas.

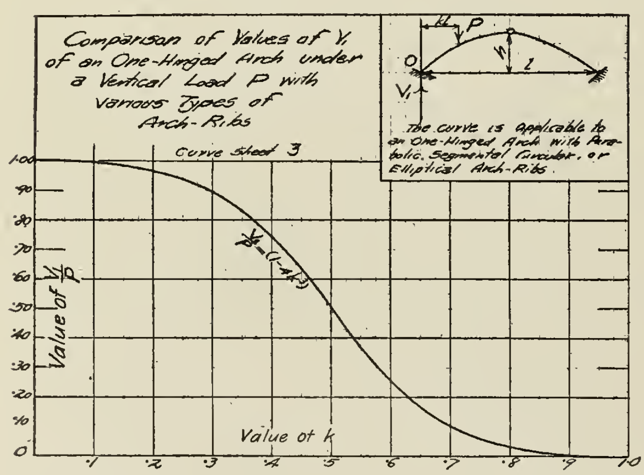
No- and one-hinged arches are similar in one respect; that is, they are both subject to vertical and horizontal reactions and have end moments under either vertical or horizontal loads. Two- and three-hinged arches are all subject to vertical and horizontal reactions, the vertical reactions being the same for those two types of arches under either vertical or horizontal loads. Temperature and rib-shortening cause horizontal reactions and the end moments in both the one and no-hinged arches, while in two-hinged arches they cause horizontal reactions but no end moments. It is generally supposed that a three-hinged arch is not subject to stresses due to a change in temperature. Strictly, however, such stresses will occur, for a fall in temperature causes a deflection of the crown hinge, and as the span does not change, the horizontal thrust will be in-

creased. The stresses produced, however, are very small.

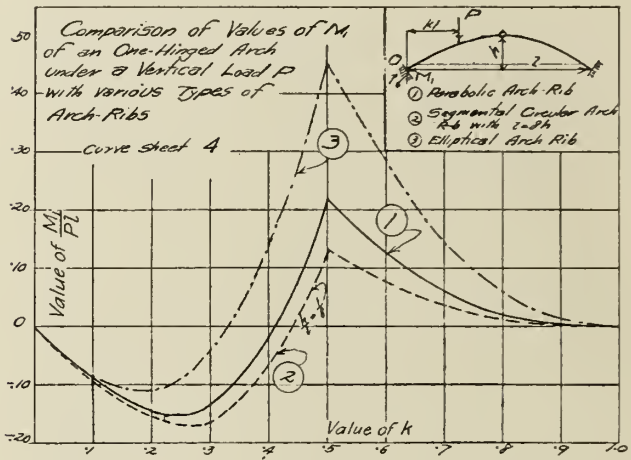


Neglecting the effect of axial thrust upon the rise and fall of temperature, the relative importance of temperature effects upon the horizontal reactions and end moments of no, one-, two-, and three-hinged arches are as follows:

Arch		No-Hinged	One-Hinged	Two-Hinged	Three-Hinged
ELLIPTICAL					
	H_1	160/8	67.2/8	12/8	0
	M_1	125.6/8	67.2/8	0	0
PARABOLIC					
	H_1	90/8	60/8	15/8	.....
	M_1	60/8	60/8	0	0

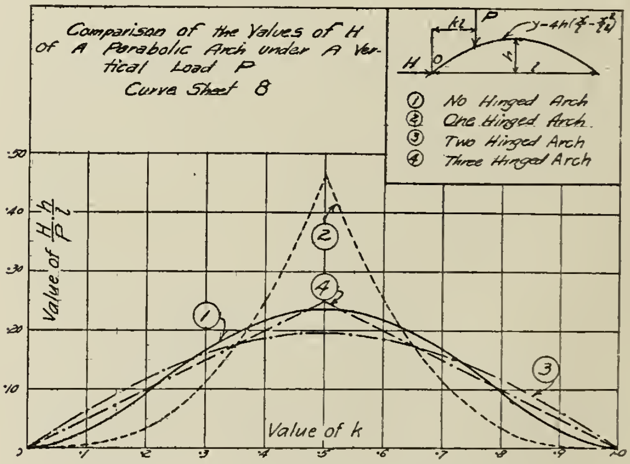


Thus, the temperature effect upon the no-hinged arch is found to be the greatest of all. Referring to H in the parabolic form, it is one and one-half times as large as for the one-hinged arch, and six times as large as for the two-hinged arch. In the elliptical type it is 2.38 times as large as that for the one-hinged arch, and 13.33 times as large as for the two-hinged arch. The temperature effect upon H of the one-hinged arch of the parabolic form is four times as large as for the two-hinged arch, while for the elliptical form it is 5.6 times as large. The temperature effect upon M of the no- and one-hinged arches is the same in the parabolic form. In the elliptical form, the former is 18.7 times as great as the latter. In the segmental circular form, the temperature effect upon H in the one-hinged arch is three times as great as that for the two-hinged arch.



With respect to the effect of rib-shortening the relative ratios above stated apply to different types of arches equally well.

So far as the formulas for the reactions are concerned, there is a decided advantage gained by those for the parabolic form over all other types of arches because of their simplicity.

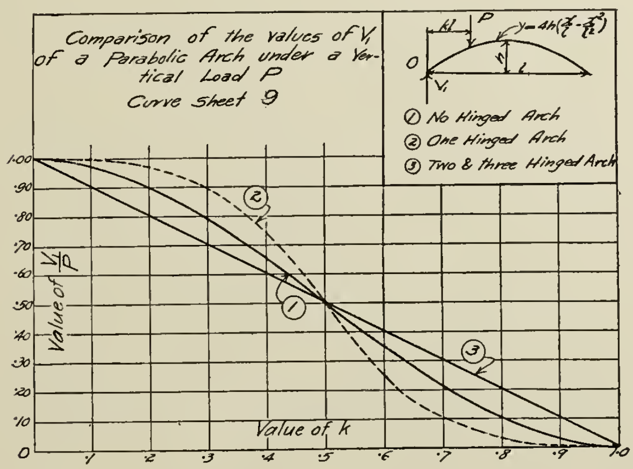


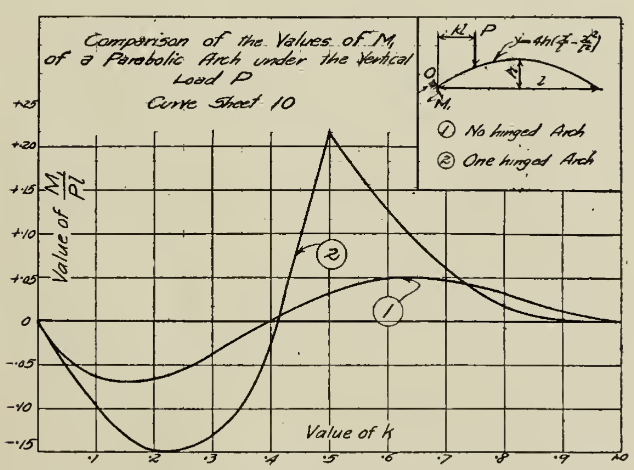
Discussion on Reaction Influence Lines

In order to study the variation of the reactions and the end moments as a single load moves over the span, reaction influence lines are drawn and compared. (See diagrams 2, 3, 4, 8, 9, and 10). Only a few of the numerous diagrams are reproduced in the abstract of the thesis.

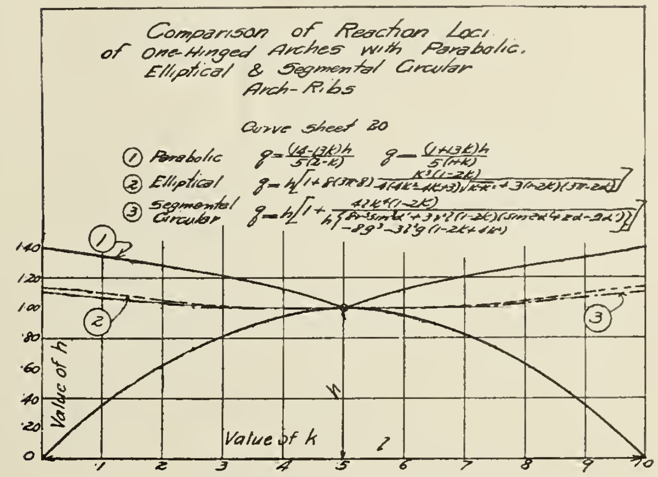
Under vertical loading, the elliptical form of the one-hinged arch has greater horizontal reactions and end moments than the parabolic and segemental circular forms. The last two forms cause less difference in the H_1 and M_1. The vertical reactions are all the same in the one-hinged arch for the three forms of ribs. There is not much difference shown under the horizontal loading. The circular form seems to have the greatest advantage for the one-

hinged arch judging from a comparison of the curves. However, the curves for the circular form are only drawn for a single case with a ratio of rise to span equal to 1-8. If other ratios were used, the results may be different. On the other hand, the parabolic form has the advantage of low reactions besides that of simplicity in formulas. Therefore, the author comes to the conclusion that the parabolic curve is the best form for the neutral axis of the arch-rib to be used for a one-hinged arch.

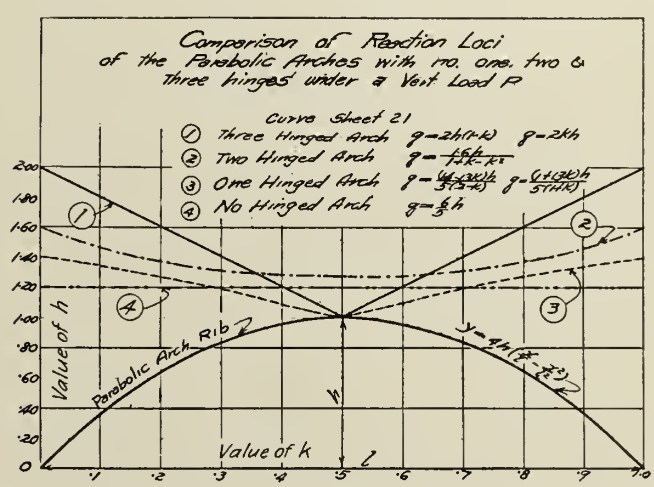




Under vertical loading the reaction influence line of H for a three-hinged arch consists of straight lines; for a two-hinged arch, a curve resembling a parabola; for a no- and a one-hinged arch, curves symmetrical at the center. Curves for one- and three-hinged arches break at the crown while those for no— and two-hinged arches do not. This is because the hinge at the crown of the one- and three-hinged arches breaks the continuity of the rib. Curves of no-, two-, and three-hinged arches are very close to each other, while that for the one-hinged arch has a great variation, being lower at the supports and higher near the crown. This brings out one disadvantageous fact for the one-hinged arch. When the load is at the crown, the value of H is about two times as much as that for the other types. This means that the supports have to resist a greater horizontal thrust, and extra care has to be taken in making the end supports secure. The end moment of the one-hinged arch has a greater variation than that of the no-hinged arch.



The one-hinged arch has an advantage over the no-hinged arch with regard to the effect of temperature and thrust, but it is not so favourable with regard to the reactions and the end moments under vertical loads. For reactions due to temperature, thrust and vertical loading, the one-hinged arch is not as advantageous as the two— and three-hinged arches. So far as the simplicity of the formulas for



the reactions is concerned, the same statement holds true. However, attention will be called subsequently to some decided advantages of the one-hinged arch.

Discussion On Reaction Loci

In order to compare the reaction loci of different types of arches, the equations of the reaction loci for no-, one-, two-, and three-hinged arches are plotted for different forms of arch ribs. They are shown in diagrams 20 and 21.

All reaction loci for the one-hinged arch with different forms of arch-ribs pass through the crown hinge. When the load is at the crown, the reaction lines must pass through the crown hinge, for otherwise there would be rotation at the crown. All reaction loci break their continuity at the crown but the two halves are symmetrical. The parabolic one-hinged arch has the highest ordinate when the load

is near the support, while the segmental circular form has the least ordinate.

For a parabolic arch-rib, the reaction locus of the three-hinged arch is a straight line for each half of the span; that for the two-hinged arch, a curve like a parabola; that for a no-hinged arch, a horizontal straight line at an ordinate of 1.2 h; and that for the one-hinged arch, a curve breaking at the center. Curves for no- and two-hinged arches are continuous, while those for the other two break at the center. Curves for no- and two-hinged arches do not pass through the crown, while those for the one- and three-hinged arches must do so. At the support, the three-hinged arch has a maximum ordinate of 2.00 h; the two-hinged arch, 1.60 h; the one-hinged arch, 1.40 h; and the no-hinged arch, 1.20 h. At the crown the curve for the two-hinged arch has the highest ordinate.

Similar statements can be drawn for the reaction loci for the elliptical forms of arches. The ordinates are highest in certain cases and lower in others, while the general forms are about the same.

Discussion On Reaction Envelop

By means of the reaction locus and the reaction lines, the position of live loading for the maximum positive and negative moments and maximum positive and negative shears for the one-, two-, three, and no-hinged arches can be found graphically. In the three- and two-hinged arches, the reaction lines can easily be drawn as soon as the position of the live load is given; because the hinges at each support fix the direction of the reaction lines. In the one- and no-hinged arches, the reaction lines cannot be so easily drawn, if the position of the load is only given; for the fixing of the supports makes the direction of the reaction lines uncertain. In order to avoid the difficulty, reaction envelops are required for both the one- and no-hinged arches.

The reaction envelop of the no-hinged arch was found by A. V. Saph, a graduate of Cornell University, to be two hyperbolas symmetrical and tangent to each other at the center at a point 2-3 h above the level of the supports. The reaction envelop of the one-hinged arch was found by the author to consist of two curves very similar to those for the no-hinged arch in form yet quite different in their equations. The curves are symmetrical and meet each other at the crown hinge. The reaction envelop is a plane curve of the third degree; because the formulas of the reactions of the one-hinged arch involves the k's in the fourth power. In the case of the no-hinged arch, the k's in the formulas of the reactions are of third power; hence a second-degree curve is obtained for the envelop.

The method of finding the positions of live loading is the same for the four types of arches, except that the reaction envelops of the no- and one-hinged arches have the function of the end hinges in the case of the three- and two-hinged arches. A comparison of the reaction envelops of the no- and one-hinged arches is given in diagram 28.

Discussion On Design With the Moment of Inertia Varying As the Angle of Inclination of the Arch-Rib

There is no marked difference in the procedure of designing the arch-rib for the two-, one-, and no-hinged steel arches with the assumption that I varies as $\sec\theta$. The procedure may be generalized in the following heads: (1) calculation of reactions and end moments from the formulas for a unit load at each panel point; (2) to find the position of live loading for maximum positive and negative moments, by either algebraic or graphical methods; (3) calculation of dead load and live load moments for each section; (4) calculation of dead load and live load thrust; (5) calculation of moments and thrust due to temperature effect by assuming a certain moment of inertia at the crown; (6) calculation of moments and thrust due to effect of rib-shortening by means of the assumed moment of inertia; (7) design of the flange area of the crown section using the maximum moments and thrusts so obtained; (8) test the moment of inertia of the crown section, and see if the assumed value of the moment of inertia at the crown section agrees with the value obtained; (9) after the right value of I_0 is obtained the flange areas at other sections can be calculated by using the relation that $I = I_t \sec\theta$; (10) the position of live load for maximum positive and negative shear is obtained by either graphical or algebraic method, the maximum shear at each section is secured by combining the dead load, live load, temperature and rib-shortening shears, and the webs are designed accordingly. The design of the three-hinged arch may be carried out in the similar order with the exception that no trial is required in securing the sections.

There is an interesting fact found in combining the moments and thrusts caused by the dead load, live load, temperature and rib-shortening to produce the maximum stress on the sections of the one-hinged arch. The live load may be considered in three ways; (1) loading for maximum positive moments; (2) loading for the maximum negative moments; (3) loading for maximum thrust. In the design with the effect of rib-shortening neglected, case (1) controls for the section 0-4 and case (3) controls for the section 5-10. In the design with the effect of the rib-shortening included, case (2) controls for the section 0-4, while case (3) still controls for the sections 5-10. The reason is quite clear; for in the sections near the center, the moments are comparatively small, while the thrusts are large. This means that a full live load is required to design the sections near the center. As there exist large bending moments in the sections near the support, naturally the moments control

the design for those sections. The effect of rib-shortening is to produce the negative moments and thrusts. This is why the negative moments control the end sections when the effect of the rib-shortening is included. The design of the one-hinged arch is different from that of the two-, and no-hinged arch in two respects. The design of the center sections of the one-hinged arch is controlled by the thrust while that of the two- and no-hinged arches is controlled by the moments. The reason is because the no-hinged and two-hinged arches have large moments at the middle sections, while the one-hinged arch has large thrusts. The signs of the moments and thrusts produced by the effect of temperature and rib-shortening are the same for all sections of the one-hinged arch, while those for the no-hinged arch are the same for the sections below 2-3 h of the rib, and opposite each other for the sections above 2-3 h of the rib. The signs of the moments and thrusts produced by the effect of temperature and rib-shortening are all opposite for the two-hinged arches.

In designing the sections at the crown, only thrust is used for the one-hinged arch. This makes the design of the one-hinged arch easier, because the right value of the moment of inertia can be secured immediately.

Positive shear governs the design of the web for all types of arches, whether the effect of rib-shortening is included or not. The latter which tends to produce negative thrust, tends to increase the positive shear in all sections for the one-hinged arch.

Under maximum loading the comparative effects of dead load, live load, temperature and rib-shortening on the flanges are shown in the following table and also on curve sheet 22:

Section	Dead load Per Cent.	Live load Per Cent.	Temperature Per Cent.	Rib-shortening Per Cent.
0	31.5	29.7	24.2	14.6
1	38.8	24.2	23.0	14.0
2	47.4	17.7	21.7	13.2
3	58.2	10.6	19.3	11.9
4	72.7	1.3	16.1	9.9
5	75.8	23.7	20.6	—20.1
6	75.9	23.8	15.5	—15.1
7	76.0	23.8	11.4	—11.1
8	76.0	23.9	8.5	— 8.4
9	76.0	23.8	6.6	— 6.5
10	75.8	23.8	6.3	— 5.9

It is seen from the curve that the dead load has the greatest effect of all, and its effect on the sections near the center is greater than those near the ends. In the first place, this is because the dead panel load is comparatively great, and naturally it takes greater stress. In the second place, the effect of temperature and rib-shortening near the center sections is greater than their effect on the sections near the ends. This makes the percentage of the stress carried by the dead load gradually decrease in the sections near the ends. The live load is the next important factor in causing flange stresses. The effect of temperature and rib-shortening can never be omitted. They have about the same effect. Temperature has the greater effect in the sections near the ends than in those near the center. The same is true for the rib-shortening. The negative signs of the rib-shortening in the sections near the ends are explained by the fact that both negative thrusts and moments are used.

Under maximum loading the comparative effects of moments and thrusts on the flanges is shown in the following table and also shown on curve sheet 23:—

Section	Moment Per Cent.	Thrust Per Cent.
0	73.1	26.9
1	67.4	32.6
2	60.3	39.7
3	51.8	48.2
4	43.1	56.9
5	2.5	97.5
6	1.5	98.5
7	.7	99.3
8	.4	99.6
9	.1	99.9
10	.0	100.0

One apparent conclusion can be drawn from the curve; that is, the moment has far greater effect on the sections near the ends than the thrust, while the thrust has far larger effect on the sections near the middle than the moment. The effect of moment on the sections 5-10 is practically nothing. As a whole, the thrust is more important than the moment. This peculiar fact is a special feature of the one-hinged arch.

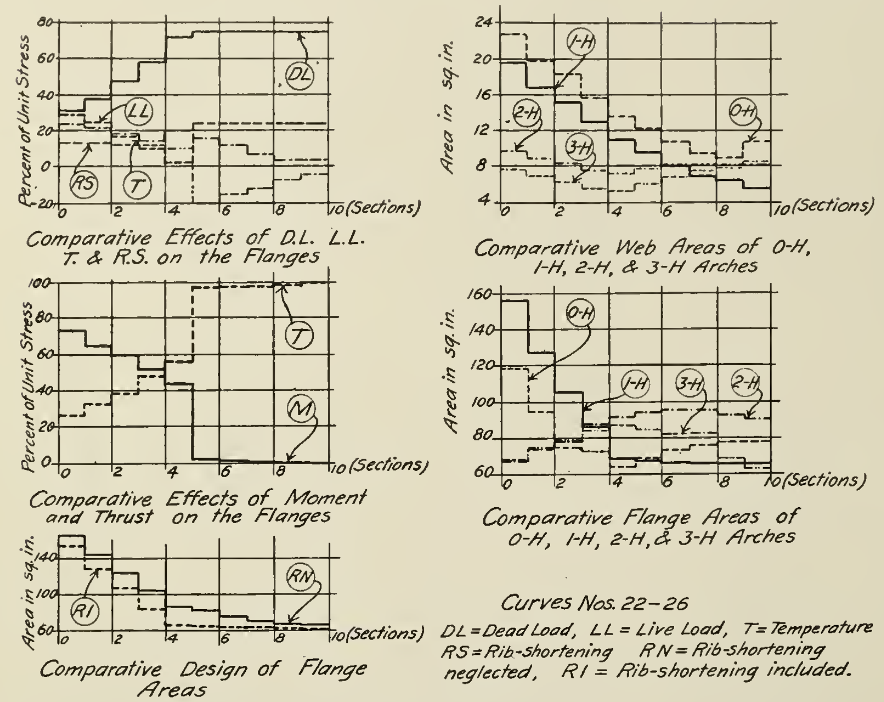
The comparative areas of the designs with the assumption $I \simeq \sec\theta$ with rib-shortening included and neglected are shown in the following table and on curve sheet 24:—

Section.	Rib-Shortening Neglected. Area in sq. in.	Rib-Shortening Inc. Area in sq. in.
0	166.69	153.70
1	142.77	127.00
2	121.39	103.60
3	105.00	84.00
4	88.84	67.75
5	82.15	67.70
6	77.39	66.80
7	73.72	66.00
8	71.10	65.45
9	69.38	64.95
10	68.71	64.71

It is seen from the curve that the one-hinged arch requires far larger areas in the sections near the ends than the sections near the crown. This is

due to the presence of the large end moments. It is also seen that the effect of rib-shortening is to decrease the areas of sections. We thus see that for this particular rise of span, the effect of rib-shortening can be safely but not economically neglected in the design. The author believes that this is equally true for other ratios of rise to span, because in any

So far as the areas of sections in the flanges are concerned, the three-hinged arch is the most favorable of all, while the no-hinged and one-hinged arch require the largest areas. The areas of the two-hinged arch are intermediate between them. The three-hinged arch requires the largest area at the quarter points; for the moments at those sec-



Comparative Effects of D.L. L.L. T. & R.S. on the Flanges

Comparative Web Areas of O-H, 1-H, 2-H, & 3-H Arches

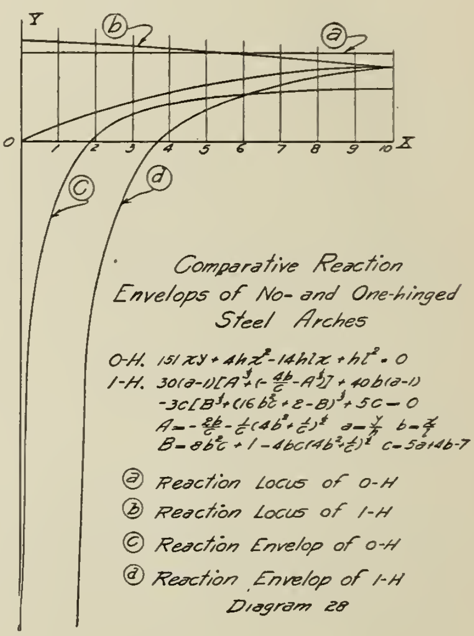
Comparative Effects of Moment and Thrust on the Flanges

Comparative Flange Areas of O-H, 1-H, 2-H, & 3-H Arches

Comparative Design of Flange Areas

Curves Nos. 22–26

DL = Dead Load, LL = Live Load, T = Temperature, RS = Rib-Shortening, RN = Rib-Shortening neglected, RI = Rib-shortening included.



Comparative Reaction Envelops of No- and One-hinged Steel Arches

(a) Reaction Locus of O-H

(b) Reaction Locus of 1-H

(c) Reaction Envelop of O-H

(d) Reaction Envelop of 1-H

Diagram 28

case the moments and thrust produced by the rib-shortening always have the same negative sign in the case of a one-hinged arch. This naturally tends to decrease the area of section required, if the area of section is controlled by the positive moments and thrust, in the design, as is usually the case in the one-hinged arch.

The comparative flange areas required for the three-, two-, one-, and no-hinged arches for the same form of the arch-rib under the same specifications are shown in the following table and on curve sheet 25. The areas given for the three-hinged arch are exact, while those for the two-, one-, and no-hinged arches are obtained from a design with the assumption $I \simeq \sec\theta$ with the effect of rib-shortening included.

Section.	Three-Hinged. sq. in.	Two-Hinged. sq. in.	One-Hinged. sq. in.	No-Hinged. sq. in.
0	68.0	68.4	153.7	119.5
1	73.0	72.8	127.0	93.1
2	78.2	78.1	103.6	75.9
3	82.4	85.0	84.0	72.3
4	84.4	90.7	67.7	65.0
5	84.1	93.1	67.7	67.9
6	81.2	94.9	66.8	72.8
7	77.7	94.8	66.0	76.9
8	71.0	92.3	65.4	78.4
9	65.0	89.3	64.9	78.3
10	64.0	89.0	64.7	77.3

tions are the largest of all. The two-hinged arch requires the largest area in the sections near the middle for the same reason. The one-hinged and no-hinged arches require the largest areas in the sections near the ends, because the end moment is the dominating factor in those sections. In comparing the areas in flanges of the one- and no-hinged arches, we see that the one-hinged arch requires comparatively large areas in the sections near the ends and small areas in the sections near the crown. The areas near the crown of the one-hinged arch are rather uniform, because the thrusts at these points are large and about equal in magnitude.

The comparative theoretical web areas required for the three-, two-, one- and no-hinged arches with the same form of the rib and designed under the same specifications are shown in the following table and also in curve sheet 26:—

Section.	Three-Hinged. sq. in.	Two-Hinged. sq. in.	One-Hinged. sq. in.	No-Hinged. sq. in.
1	7.91	10.04	19.59	22.83
2	6.95	8.98	16.76	19.88
3	6.31	8.20	14.95	18.22
4	5.77	7.67	13.03	15.51
5	5.45	7.45	11.10	13.58
6	6.10	7.61	9.55	12.08
7	6.95	7.92	7.95	10.87
8	7.61	8.11	6.89	9.70
9	7.97	8.12	6.45	9.18
10	8.13	7.96	5.75	10.46

It is seen that the no-hinged arch requires the largest web area, while the three-hinged arch, the least of all. The one-hinged arch requires smaller web areas than the no-hinged arch, while the two-hinged arch requires greater areas than the three-hinged arch. The no-hinged and one-hinged arches require larger web areas in the sections near the ends and smaller areas in the sections near the crown. This is due to the fact that the shear in the no-hinged and one-hinged arches is far greater near the ends than in the sections near the crown. On the other hand, the web areas of the three-, and two-hinged arches are rather uniform, since the shears in the sections of the three- and two-hinged arches are rather uniform. The marked difference is caused by the fixing of the supports in the one case, and the hinging of the supports in the other.

The following table shows the results of the design based upon the assumption $I \approx \sec\theta$

Section.	Composition	Moment of Inertia.	Ratio of I.	Sec θ.	Proposed Assumption of I.
0-2	6∠s 6″ × 6″ × 9/16″ 3 Pls 14″ × 9/16″ 5 Pls 18″ × 7/8″ 1 Pls 18″ × 3/4″	286,000	2.49	1.07	250%
2-4	6∠s 6″ × 6″ × 9/16″ 3 Pls 14″ × 9/16″ 2 Pls 18″ × 3/4″ 1 Pls 18″ × 13/16″	190,600	1.59	1.05	160%
4-6	6∠s 6″ × 6″ × 9/16″ 3 Pls 14″ × 9/16″ 3 Pls 18″ × 5/16″	123,800	1.03	1.03	103%
6-8	6∠s 6″ × 6″ × 9/16″ 3 Pls 14″ × 9/16″ 1 Pls 18″ × 1/4″	121,700	1.02	1.01	101%
8-10	6∠s 6″ ×6″ × 9/16″ 1 Pls 14″ × 9/16″ 1 Pls 18″ × 3/16″	119,700	1.00	1.00	100%

It is seen that the assumption $I \approx \sec\theta$ is about right for the sections near the crown, and is in great error for the sections near the ends.

DISCUSSION ON TRUE DESIGN

So far as the reactions are concerned, the effect of final design as compared with that of the preliminary design, assuming $I \approx \sec\theta$, is to increase the horizontal reactions when the load is near the center of the span and to decrease the same when the load is near the ends. There is no appreciable change in the vertical reactions for the two designs. The end moments have been affected so greatly that under the full loading, a negative end moment is produced in this design, which was not the case in the preliminary design. The latter has a serious effect in designing the sections, because the sign of the dead load moment is changed to negative, and the maximum moments at various sections are thereby changed.

The preliminary design by assuming $I \approx \sec\theta$ is far from being correct. It makes the flange areas at the various sections far different from the true values. The error is on the unsafe side. It is necessary that a revised design be made in designing a one-hinged arch.

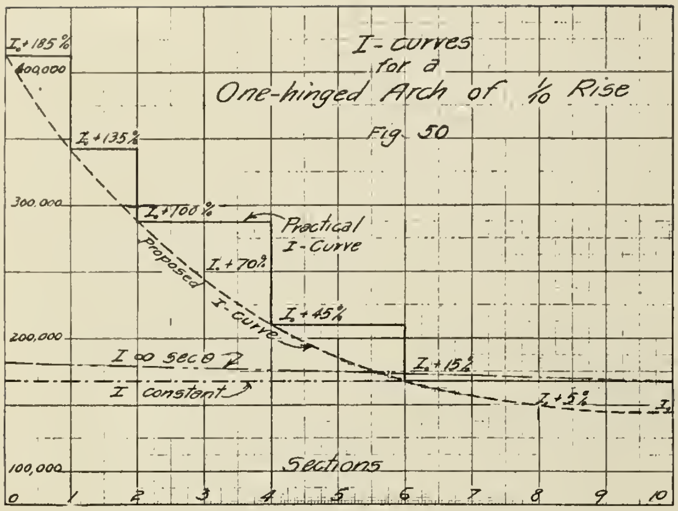
The preliminary design by assuming $I \approx \sec\theta$ is nearest to the true value for the two-hinged arch, while it differs most in the one-hinged arch. The relative error of the preliminary design from the true design of the no-, one-, and two-hinged arches with same dimensions and designed to carry the same load is shown below:

Section	No-hinged per cent	One-hinged per cent	Two-hinged per cent
0	—15.5	—30.7	5.2
1	—13.4	—31.6	—3.7
2	— 7.6	—33.1	—4.7
3	— 2.4	—34.4	—2.5
4	— 3.6	—40.5	2.4
5	8.7	—33.3	3.3
6	5.6	—27.0	7.6
7	4.1	—21.0	12.8
8	3.7	—16.1	12.1
9	2.5	—12.8	11.6
10	2.1	—11.7	13.7

Thus, we see that the error in the one-hinged arch is the largest of all. This is due to the presence of the large moments in the sections near the ends and the large thrust in the sections near the crown. The variation of the moments and thrusts in the sections is such that the moment of inertia in the various sections required is not in accordance with the relation $I \approx \sec\theta$.

As the assumption $I \approx \sec\theta$ is far from being true, the I-Curve for a one-hinged arch with 1/10 rise is recommended by the author, by means of which a closer result can be secured in the approximate design. A similar curve was recommended for the no-hinged arch by P. H. Chen in his thesis above named. The comparative values are shown below:

Section	One-hinged	No-hinged
0	I_0 + 185%	I_0 + 120%
1	I_0 + 135%	I_0 + 60%
2	I_0 + 100%	I_0 + 30%
3	I_0 + 70%	I_0 + 15%
4	I_0 + 45%	I_0
6	I_0 + 15%	I_0 + 10%
8	I_0 + 5%	I_0 — 10%
10	I_0	I_0



It is seen from the curves that the assumptions of $I \propto sec.^{\theta}$ and $I =$ constant are nearly correct in the sections near the center, while they are too small in the sections near the ends. The exceedingly large moments near the ends both in the no-hinged and one-hinged arches necessitate the use of the large sections and accordingly the values of the large moments of inertia.

DISCUSSION OF METHODS OF DEFLECTION COMPUTATIONS

Deflections in the one-hinged arch-ribs are contributed by two factors, that due to thrust and that due to moment. The methods used by the author in finding these deflections are rather interesting and are here given.

(a) DEFLECTION DUE TO MOMENTS—The formulas for finding the horizontal and vertical deflections of a curved beam were given in equations (d) and (e) respectively. As the continuity of the one-hinged arch is broken at the crown by the presence of the hinge, the formulas could not be applied directly to the entire span. A method of separating the arch into two cantilevers is introduced (Fig. 3). For example, when the load is at a certain point on the left half of the span, the reactions acting on the left cantilever are H_1 and V_2, while those on the right cantilever are H_1 and V_2. lever are contributed by three factors, due to P, H_1 and V_2, while those in the right cantilever are contributed by two factors, H_1 and V_2. These factors can be easily found by considering each half in turn as a cantilever with one load on the arch. Thus, under the vertical load P on the left cantilever, we only need to find the deflections in the cantilever due to a hortizontal load at the free end, a vertical load at the free end, and a vertical applied load. The deflections due to H_1 and V_2 in the left half of the arch are just the same in magnitude as those due to H_1 and V_2 in the right half of the arch. The final value of the vertical or horizontal deflections can be easily obtained by taking the algebraic sum of the vertical or horizontal deflections contributed by the various factors.

If the deflection diagrams are required for a unit load at different positions on the left half of the span, the method is exceedingly simple. We need only to draw the deflection diagrams of the cantilever with a unit horizontal load at the free end, and a unit vertical load at different positions on the cantilever corresponding to the positions on the arch. These are obtained as unit deflections. The values of H_1 and V_2 are then found for different positions of the unit applied load. The deflections due to H_1 and V_2 are then found by multiplying the unit deflections by the values of H_1 and V_2. Using proper signs of deflections contributed by each factor and taking the algebraic sum, the deflections at various points on the arch can be found for different positions of the loading.

The method has several advantages: first, it is simple as well as easy; second, it offers the opportunity of studying the deflections contributed by each factor; third, it is easy to discover mistakes. By means of the graphical method, the deflection on the cantilever due to a unit vertical or horizontal load at any position on the beam can be easily calculated. By arranging the work in a systematic way, errors can be easily discovered by comparison.

The method is applicable in any of the four cases: (1) vertical deflection under the vertical load, (2) horizontal deflection under the vertical load, (3) vertical deflection under the horizontal load, (4) horizontal deflection under the horizontal load. The only difference lies in the magnitude and direction of H_1 and V_2 and the unit horizontal or vertical deflections to be found in the cantilevers.

(b) DEFLECTION DUE TO THRUST—The general formula for the deflection due to thrust is given by

$$\text{Deflection} = \int_a^b Tt\frac{ds}{AE}$$

in which T is the thrust due to the applied load P; while small t is the thrust due to a load unity P'' applied at the point whose deflection is sought, the direction of P″ being the same as the direction of of the deflection required. Let H' and V' be the horizontal and vertical shear (not in the normal section) immediately on the left of the section considered; and H'' and V'', those due to P'' respectively. Then, by substitution, the general formula of deflection due to thrust becomes,

$$\text{Deflection} = \int_a^b HH''\cos^2\theta\frac{ds}{AE} + \int_a^b VV''\sin^2\theta\frac{ds}{AE}$$

which is applicable to the four cases above named, that is, the vertical and horizontal deflections due to the horizontal load.

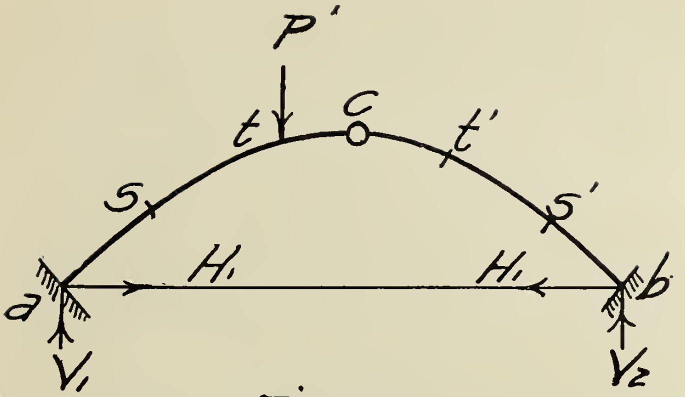


Fig. 59

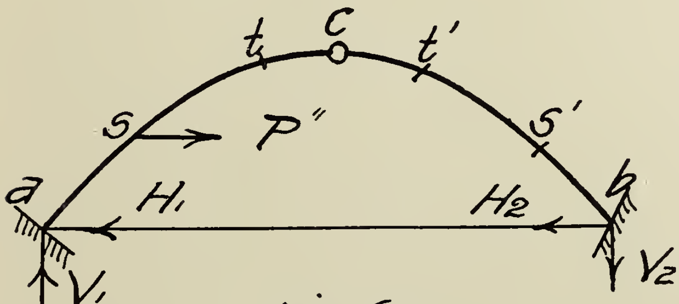


Fig 60

In order to simplify the numerical work(the above formula may be transformed into a simpler form for each of the different cases. For example, let us take the case of the horizontal deflection under the vertical loading. Let a unit load P' be applied at t and the horizontal deflection at s be found. (Figs. 59 and 60). The corresponding points on the other half of the arch are called s' and t' In examining the loading and reactions closely, we find that H' and H'' in s-c and s'-c are correspondingly equal. H' in a-s is also equal to H' in b-s' under the load P'; while under the load P''', H' in a-s is equal to H_1 and H' in b-s' is equal to H_2. Also V'' in t-c and t'-c are correspondingly equal. V'' in a-t is equal to V'' in t'-b, while V'' in a-$t$$=V_1$ and V'' in b-t is equal to V_2. Using the proper signs, the formula is given by

$$D=2\int_s^c H'H''\cos^2\theta\frac{ds}{AE}-\int_a^s(H_1''-H_2'')H'\cos^2\theta\frac{ds}{AE}$$
$$+\int_a^t(V_1'V_2'')V'\sin^2\theta\frac{ds}{AE}-2\int_t^c V'V''\sin^2\theta\frac{ds}{AE}$$

It is seen that the terms containing *Vs can be* neglected, because the angle of is not greater than 30 degrees and the squares of sines will be small in value. The net result of the horizontal deflection due to *Vs* is negligible in comparison with the deflections caused by *Hs* in the thrusts and that by the moments. By means of transformations the following formulas are derived with several approximations:

For vertical deflection due to thrust under the vertical load;

$$D=2\int_a^c H'H''\cos^2\theta\frac{ds}{AE}$$

for horizontal deflection due to thrust under the horizontal load;

$$D=2\int_t^c H'H''\cos^2\theta\frac{ds}{AE}-\int_a^s(H_1'H_1''-H_2'H_2'')\cos^2\theta\frac{ds}{AE}$$
$$-\int_s^t H_2''(H_1'-H')$$

and for vertical deflection due to thrust under the horizontal load;

$$D=2\int_t^c H'H''\cos^2\theta\frac{ds}{AE}-\int_a^t(H_1''-H_2'')H'\cos^2\theta\frac{ds}{AE}$$

DISCUSSION ON DEFLECTIONS

The moments and thrusts are equally important in causing the deflections in the one-hinged arch. The latter is especially important in the sections near the center. In considering the deflections caused by the thrust, the vertical forces may be neglected without appreciable error. Shear can be neglected too.

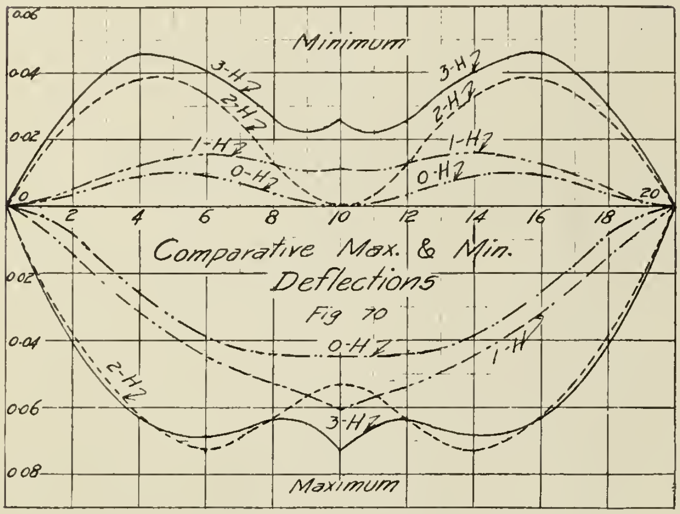
Under the vertical loading the vertical deflections near the end of the arch are similar to those of the no-hinged arch, while those near the center resemble in the form those of the three-hinged arch. As a whole, the one-hinged arch is subject to greater vertical deflection than the no-hinged arch and less vertical deflections than the two-, and three-hinged arches for the vertical loading. Thus, the no-hinged arch is the most favorable in stiffness among the four types of arches, so far as vertical deflections under the vertical load are concerned. The one-hinged arch comes the next, the two-hinged arch comes third, and the three-hinged arch is the most unfavorable of all. Comparative deflection curves for the four types of arches with the same dimensions and designed under the same loading are prepared by the author for different positions of the loading. They are not here reproduced due to the limit of space.

The same relation is found for the horizontal deflections under the vertical loading for the four types of arches. The horizontal deflections of the three- and two-hinged arches are much greater than those of the no- and one-hinged arches, while the horizontal deflections of the no- and one-hinged arches are about the same. Comparative curves of horizontal deflections under the vertical load for the four types of arches are also prepared by the author.

Under the horizontal loading, the vertical and horizontal deflections for the one-hinged arch are found to be very small. This seems to indicate that the vibrations under the moving train are low, and the effect of wind is also rather unimportant for this type of the arch. Comparative curves of deflections under the horizontal loading for the four types of the arches shows that the one-hinged arch has greater horizontal and vertical deflections than the no-hinged arch, and less vertical and horizontal

deflections than the two- and three-hinged arches.

The comparative maximum and minimum deflections for the four types of arches are shown in Fig. 70. The one-hinged arch is again seen to be stiffer than the two- and three-hinged arches, and not so stiff as the no-hinged arch.



The yielding of a support has a serious effect on the stresses. With a horizontal movement of the support of one inch away from the original position, an additional value of 126.8 kips is increased in the horizontal reactions. This increases the thrust at the crown to twice the original value, and thus greatly affects the safety of the arch. The yielding of supports has a similar effect on the two- and no-hinged arches. The effect on the one-hinged arch is a little less than that on the no-hinged arch, and much greater than that in the two-hinged arch. For the same amount of horizontal movement, the increase in the horizontal reaction in the two-hinged arch is 29.8 kips, while that in the no-hinged arch is 161.0 kips. Thus, we see that a perfectly firm foundation is required for these three types of arches.

Remarks

The main feature of this investigation consists in the discovering new formulas and relations for the one-hinged arch. These formulas and relations are then carefully studied with the aid of numerous curves and diagrams, and their characteristics are revealed by means of critical comparisons. Although the main purpose of this thesis consists in searching for new theoretical facts, it is hoped that it may stimulate investigations in a practical treatment of the one-hinged arch with regard to its design and construction. The knowledge of the subject, however, is still in its infancy. Notwithstanding the large amount of study and the labor devoted to the subject treated in this thesis, other subjects remain to be investigated, such as the economic ratio of rise to span, the relative merits of solid and open webs, the secondary stresses, etc.